VII. *On the Theory of Definite Integrals.* *By* W. H. L. Russell, *Esq.*, *B.A.* *Communicated by* A. Cayley, *Esq.*, *F.R.S.*

Received October 30,—Read November 23, 1854.

I PROPOSE in the following paper to investigate some new methods for summing various kinds of series, including almost all of the more important which are met with in analysis, by means of definite integrals, and to apply the same to the evaluation of a large number of definite integrals. In a paper which appeared in the Cambridge and Dublin Mathematical Journal for May 1854, I applied certain of these series to the integration of linear differential equations by means of definite integrals. Now Professor Boole has shown, in an admirable memoir which appeared in the Philosophical Transactions for the year 1844, that the methods which he has invented for the integration of linear differential equations in finite terms, lead to the summation of numerous series of an exactly similar nature, whence it follows that the combination of his methods of summation with mine, will lead to the evaluation of a large number of definite integrals, as will be shown in this paper. It is hence evident that the discovery of other modes of summing these series by means of definite integrals must in all cases lead to the evaluation of new groups of definite integrals, as will also be shown in the following pages. I then point out that these investigations are equivalent to finding all the more important definite integrals whose values can be obtained in finite terms by the solution of linear differential equations with variable coefficients. Again, there are certain algebraical equations which can be solved at once by Lagrange's series, and by common algebraical processes; the summation of the former by means of definite integrals affords us a new class of results, which I next consider. A continental mathematician, M. Smaasen, has given, in a recent volume of Crelle's Journal, certain methods of combining series together which give us the means of reducing various multiple integrals to single ones. The series hitherto considered are what have been denominated "factorial series"; but, lastly, I proceed to show that analogous processes extend to series of a very complicated nature and of an entirely different form, and for that purpose sum by means of definite integrals certain series whose values are obtained in finite terms in the 'Exercices des Mathématiques' by means of the Residual Calculus. The total result will be the evaluation of an enormous number of definite integrals on an entirely new type, and the application of definite integrals to the summation of many intricate series.

Let us first consider the series whose general term is

$$\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{\beta(\beta+1)\ldots(\beta+n-1)}\cdot\frac{\alpha'(\alpha'+1)\ldots(\alpha'+n-1)}{\beta'(\beta'+1)\ldots.(\beta'+n-1)}\cdot\frac{x^n}{1.2.3\ldots n}.$$

Its sum will be found to be

$$\frac{\Gamma\beta}{\Gamma\alpha\Gamma(\beta-\alpha)}\cdot\frac{\Gamma\beta'}{\Gamma\alpha'\Gamma(\beta'-\alpha')}\cdot\cdot\int_0^1\int_0^1\ldots v^{\alpha-1}z^{\alpha'-1}\ldots(1-v)^{\beta-\alpha-1}(1-z)^{\beta'-\alpha'-1}\ldots\varepsilon^{xvz\ldots}dvdz.$$

Next, if we consider the series, whose general term is

$$\frac{1}{\beta(\beta+1)\ldots.(\beta+n-1)\ \beta'(\beta'+1)\ldots(\beta'+n-1)}\cdot\frac{x^n}{1.2.3\ldots n},$$

we find for the sum

$$\frac{\Gamma\beta.\varepsilon}{2\pi}\cdot\frac{\Gamma\beta'.\varepsilon}{2\pi}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dz\ dz'\ldots\frac{\varepsilon^{i(z+z'\ldots)}}{(1+iz)^{\beta}(1+iz')^{\beta'}\cdots}\varepsilon^{\frac{x}{(1+iz)(1+iz')\ldots.}}.$$

We may easily reduce this to a possible form by putting $z=\tan\theta$, $z'=\tan\theta'$, &c. If the series to be summed is of the nature of both the kinds of series we have been discussing, we must combine the two methods of summation together.

Now consider the following differential equation:

$$u+\phi(\mathrm{D})\varepsilon^{r\omega}u=0,\ \text{where } \varepsilon^{\omega}=x.$$

This equation can always be satisfied when the factors in the denominator of $\phi(\mathrm{D})$ are real and unequal by a series of the form

$$u=1+\frac{\alpha\beta\gamma..}{\alpha'\beta'\gamma'..}x+\frac{\alpha(\alpha+1)\beta(\beta+1)\gamma(\gamma+1)\ldots}{\alpha'(\alpha'+1)\beta'(\beta'+1)\gamma'(\gamma'+1)..}\frac{x^2}{1.2}+\ \&\text{c}.$$

We shall suppose that the number of the quantities α, β, γ &c. is always less than the number of the quantities α', β', γ' &c., and, for the present, that the magnitude of α, β, γ &c. is always less than that of α', β' &c., each to each. Then the sum of this series by means of definite integrals can always be found by the preceding theorems. Now Professor Boole has given, in the memoir I have before mentioned, the conditions which are necessary in order that the equation $u+\phi(\mathrm{D})\varepsilon^{r\omega}u=0$ may be integrable in finite terms, which are therefore the conditions that the sum of the above series, and consequently the value of any definite integral equivalent to it, may be found in finite terms. I shall now give some instances of the evaluation of definite integrals by the application of these principles. Let us consider the symbolical equation

$$u-\frac{\mu^2\varepsilon^{2\theta}u}{(\mathrm{D}-1)(\mathrm{D}-4)}=0,\ \text{where } \varepsilon^{\theta}=x,$$

and assume for its solution

$$v-\frac{\mu^2\varepsilon^{2\theta}v}{(\mathrm{D}-1)(\mathrm{D}-2)}=0,\ \text{so that } u=(\mathrm{D}-2)v,$$

whence $$v=\mathrm{C}_1x\varepsilon^{\mu x}+\mathrm{C}_2x\varepsilon^{-\mu x}.$$

Hence $$u=\mathrm{C}_1(\mu x^2-x)\varepsilon^{\mu x}+\mathrm{C}_2(\mu x^2+x)\varepsilon^{-\mu x};$$

and we find from this the series

$$x^4\left\{1+\frac{\mu^2}{\frac{5}{2}.1}\cdot\frac{x^2}{2^2}+\frac{\mu^4}{\frac{5}{2}.\frac{7}{2}.1.2}\cdot\frac{x^4}{2^4}+\&c.\right\}=\frac{3}{2\mu^3}\{(\mu x^2-x)\varepsilon^{\mu x}+(\mu x^2+x)\varepsilon^{-\mu x}\} \quad . \quad . \quad \text{(I.)}$$

Whence we find, putting μ for $\frac{\mu^2x^2}{2^2}$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\sqrt{\cos\theta}\,\varepsilon^{\mu\cos^2\theta}\cos\left(\mu\sin\theta\cos\theta+\frac{5\theta}{2}-\tan\theta\right)$$
$$=\frac{\sqrt{\pi}}{2\mu^2\varepsilon}\left\{2\mu\varepsilon^{2\sqrt{\mu}}-\sqrt{\mu}\,\varepsilon^{2\sqrt{\mu}}+2\mu\varepsilon^{-2\sqrt{\mu}}+\sqrt{\mu}\,\varepsilon^{-2\sqrt{\mu}}\right\}.$$

Next consider the symbolical equation

$$(\mathrm{D}-1)(\mathrm{D}-3)(\mathrm{D}-5)u-\mu^3\varepsilon^{3\omega}u=0,\ \text{ where } \varepsilon^{\omega}=x;$$

and assume as the transformed equation

$$(\mathrm{D}-1)(\mathrm{D}-2)(\mathrm{D}-3)v-\mu^3\varepsilon^{3\omega}v=0.$$

Then $$u=(\mathrm{D}-2)v,$$

and $$v=\mathrm{C}_1x\varepsilon^{\mu x}+\mathrm{C}_2x\varepsilon^{\mu\alpha x}+\mathrm{C}_3x\varepsilon^{\mu\beta x};$$

where 1, α, β are the three cube roots of unity.

Hence $$u=\mathrm{C}_1(\mu x^2-x)\varepsilon^{\mu x}+\mathrm{C}_2(\alpha\mu x^2-x)\varepsilon^{\alpha\mu x}+\mathrm{C}_3(\beta\mu x^2-x)\varepsilon^{\beta\mu x}.$$

We must determine C_1, C_2, C_3 according to the series we have to sum.

If $$\mathrm{C}_1=\frac{8}{3\mu^4},\quad \mathrm{C}_2=-\frac{4(1+\sqrt{-3})}{3\mu^4},\quad \mathrm{C}_3=-\frac{4(1-\sqrt{-3})}{3\mu^4},$$

we find

$$\left.\begin{array}{l}x^5\left\{1+\frac{1}{\frac{5}{3}.\frac{7}{3}.1}\cdot\frac{\mu^3x^3}{3^3}+\frac{1}{\frac{5}{3}.\frac{8}{3}.\frac{7}{3}.\frac{10}{3}.1\cdot2}\,\frac{\mu^6x^6}{3^6}+\&c.\right\}\\ \quad=\frac{8}{3\mu^4}(\mu x^2-x)\varepsilon^{\mu x}+\frac{8}{3\mu^4}(2\mu x^2+x)\varepsilon^{-\frac{\mu x}{2}}\cos\frac{\sqrt{3}}{2}\mu x\\ \quad-\frac{8\sqrt{3}}{3\mu^4}x\varepsilon^{-\frac{\mu x}{2}}\sin\frac{\sqrt{3}}{2}\mu x\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II.)}$$

Whence $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta d\phi\varepsilon^{\mu\cos\theta\cos\phi\cos(\theta+\phi)}\cos^{-\frac{1}{3}}\theta\cos^{\frac{1}{3}}\phi$$

$$\cos\left\{\mu\cos\theta\cos\phi\sin(\theta+\phi)+\frac{5\theta}{3}+\frac{7\phi}{3}-(\tan\theta+\tan\phi)\right\}$$

$$=\frac{2\pi}{3\sqrt{3}\varepsilon^2\sqrt[3]{\mu^4}}\left\{(3\sqrt[3]{\mu}-1)\varepsilon^{3\sqrt[3]{\mu}}+(6\sqrt[3]{\mu}+1)\varepsilon^{-\frac{3\sqrt[3]{\mu}}{2}}\cos\frac{3\sqrt{3}\sqrt[3]{\mu}}{2}\right.$$
$$\left.-\sqrt{3}\varepsilon^{-\frac{3\sqrt[3]{\mu}}{2}}\sin\frac{3\sqrt{3}\sqrt[3]{\mu}}{2}\right\}.$$

Also

$$\left.\begin{aligned}&x^6\left\{1+\frac{1}{\frac{7}{3}\cdot\frac{8}{3}\cdot 1}\frac{\mu^3x^3}{3^3}+\frac{1}{\frac{7}{3}\cdot\frac{10}{3}\cdot\frac{8}{3}\cdot\frac{11}{3}\cdot 1\cdot 2}\frac{\mu^6x^6}{3^6}+\&c.\right\}\\ &=\frac{40}{3\mu^5}(\mu x^2-2x)\varepsilon^{\mu x}-\frac{40x}{3\mu^5}(\mu x-2)\varepsilon^{-\frac{\mu x}{2}}\cos\frac{\sqrt{3}}{2}\mu x\\ &\qquad+\frac{40\sqrt{3}x}{3\mu^5}(\mu x+2)\varepsilon^{-\frac{\mu x}{2}}\sin\frac{\sqrt{3}}{2}\mu x\end{aligned}\right\} \quad \ldots\ldots \quad \text{(III.)}$$

Whence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta d\phi\varepsilon^{\mu\cos\theta\cos\phi\cos(\theta+\phi)}\cos^{\frac{1}{3}}\theta\cos^{\frac{2}{3}}\phi$$

$$\cos\left\{\mu\cos\theta\cos\phi\sin(\theta+\phi)+\frac{7\theta}{3}+\frac{8\phi}{3}-(\tan\theta+\tan\phi)\right\}$$

$$=\frac{2\pi}{3\sqrt{3}\varepsilon^2\sqrt[3]{\mu^5}}\left\{(3\sqrt[3]{\mu}-2)\varepsilon^{3\sqrt[3]{\mu}}-(3\sqrt[3]{\mu}-2)\varepsilon^{-\frac{3\sqrt[3]{\mu}}{2}}\cos\frac{3\sqrt{3}\sqrt[3]{\mu}}{2}\right.$$

$$\left.+\sqrt{3}(3\sqrt[3]{\mu}+2)\varepsilon^{-\frac{3\sqrt[3]{\mu}}{2}}\sin\frac{3\sqrt{3}\sqrt[3]{\mu}}{2}\right\}.$$

Again, let the symbolical equation be

$$(D-1)(D-2)(D-5)u-\mu^2(D-3)\varepsilon^{2\omega}u=0,$$

and let the transformed equation be

$$(D-1)(D-2)v-\mu^2\varepsilon^{2\omega}v=(D-1)(D-2)V,$$

whence $u=(D-3)v,\qquad 0=(D-3)V.$

Hence we find $V=Cx^3,$

and $v=C_1x+C_2x\varepsilon^{\mu x}+C_3x\varepsilon^{-\mu x},$

whence $u=-2C_1x+C_2(\mu x^2-2x)\varepsilon^{\mu x}+C_3(-\mu x^2-2x)\varepsilon^{-\mu x};$

we determine C_1, C_2, C_3 according to the series we have to sum. Hence we find

$$x^5\left\{1+\frac{2}{\frac{5}{2}.3.1}\frac{\mu^2x^2}{2^2}+\frac{2.3}{\frac{5}{2}.\frac{7}{2}.3.4.1.2}\frac{\mu^4x^4}{2^4}+\&c.\right\}=\frac{24x}{\mu^4}+\frac{6}{\mu^4}(\mu x^2-2x)\varepsilon^{\mu x}-\frac{6}{\mu^4}(\mu x^2+2x)\varepsilon^{-\mu x}. \quad \text{(IV.)}$$

Hence we have

$$\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta dv\, v(1-v)^{-\frac{1}{2}}\cos\theta\varepsilon^{\mu v\cos^2\theta}\cos(\mu v\sin\theta\cos\theta+3\theta-\tan\theta)$$

$$=\frac{2\pi}{\mu^2\varepsilon}+\frac{\pi}{\mu^2\varepsilon}(\sqrt{\mu}-1)\varepsilon^{2\sqrt{\mu}}-\frac{\pi}{\mu^2\varepsilon}(\sqrt{\mu}+1)\varepsilon^{-2\sqrt{\mu}}.$$

By a similar method we find

$$x^4\left\{1+\frac{2}{3.\frac{7}{2}.1}\frac{\mu^2x^2}{2^2}+\frac{2.3}{3.4.\frac{7}{2}.\frac{9}{2}.1.2}\frac{\mu^4x^4}{2^4}+\&c.\right\}=\frac{120x^2}{\mu^4}+\frac{30}{\mu^5}(\mu x^2-3x)\varepsilon^{\mu x}+\frac{30}{\mu^5}(\mu x^2+3x)\varepsilon^{-\mu x}, \quad \text{(V.)}$$

whence we have

$$\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta dv\, v(1-v)^{\frac{1}{2}}\cos\theta\, \varepsilon^{\mu v\cos^2\theta}\cos(\mu v\sin\theta\cos\theta+3\theta-\tan\theta)$$

$$=\frac{2\pi}{\mu^2\varepsilon}+\frac{\pi}{4\mu^{\frac{5}{2}}\varepsilon}(2\sqrt{\mu}-3)\varepsilon^{2\sqrt{\mu}}+\frac{\pi}{4\mu^{\frac{5}{2}}\varepsilon}(2\sqrt{\mu}+3)\varepsilon^{-2\sqrt{\mu}}.$$

It is to be particularly remarked, that we may in many cases simplify the final results, which we obtain by means of these summations, by the use of the theorem

$$\Gamma\frac{1}{n}\,\Gamma\frac{2}{n}\,\Gamma\frac{3}{n}\cdots\Gamma\frac{n-1}{n}=(2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}}.$$

Again, let $(D-1)(D-3)(D-5)u-\mu(D-2)(D-4)\varepsilon^{\theta}u=0$,

and assume as the transformed equation

$$(D-1)v-\mu\varepsilon^{\theta}v=0.$$

Then $u=(D-2)(D-4)v$

$$0=(D-2)(D-4)V,$$

whence $V=Ax^2+Bx^4$,

and $v=C_1(\mu x^2-x)\varepsilon^{\mu x}+C_2x^3+C_3x$,

whence $u=C_1(\mu^2x^3-3\mu x^2+3x)\varepsilon^{\mu x}+C_2x^3+C_3x$,

where the constants must be determined by comparison of this expression with the series to be summed. Thus we have

$$x^5\left\{1+\frac{2.4}{3.5}\mu x+\frac{2.3.4.5}{3.4.5.6}\,\frac{\mu^2x^2}{1.2}+\&c.\right\}=\frac{8}{\mu^4}\varepsilon^{\mu x}(\mu^2x^3-3\mu x^2+3x)-\frac{24x}{\mu^4}+\frac{4}{\mu^2}x^3. \quad . \quad . \quad \text{(VI.)}$$

Hence $\displaystyle\int_0^1\int_0^1 vz^3\varepsilon^{\mu vz}dvdz=\frac{\varepsilon^{\mu}}{\mu^4}(\mu^2-3\mu+3)-\frac{3}{\mu^4}+\frac{1}{2\mu^2}.$

Moreover we shall find

$$x^6\left\{1+\frac{2.5}{4.6}\mu x+\frac{2.3.5.6}{4.5.6.7}\cdot\frac{\mu^2x^2}{1.2}+\&c.\right\}=\frac{30}{\mu^5}(\mu^2x^3-4\mu x^2+4x)+\frac{10}{\mu^2}\left(x^4+\frac{3x^3}{\mu}\right)-\frac{120}{\mu^5}x, \quad . \quad \text{(VII.)}$$

whence $\displaystyle\int_0^1\int_0^1 vz^4(1-v)\varepsilon^{\mu vz}dvdz=\frac{\varepsilon^{\mu}}{\mu^5}(\mu-2)^2+\frac{1}{3\mu^2}\left(1+\frac{3}{\mu}\right)-\frac{4}{\mu^5}.$

We shall also find

$$x^4\left\{1+\frac{2}{4}\mu x+\frac{2.3}{4.5}\cdot\frac{\mu^2x^2}{1.2}+\frac{2.3.4}{4.5.6}\cdot\frac{\mu^3x^3}{1.2.3}+\&c.\right\}=\frac{6}{\mu^3}(\mu x^2-2x)\varepsilon^{\mu x}+\frac{6}{\mu^3}(\mu x^2+2x). \quad \text{(VIII.)}$$

Hence $\displaystyle\int_0^1 v(1-v)\varepsilon^{\mu v}dv=\frac{1}{\mu^3}(\mu-2)\varepsilon^{\mu}+\frac{1}{\mu^3}(\mu+2).$

These three last integrals can be obtained by ordinary integration. I have introduced them here partly for the sake of system, and partly because we shall require the series which they represent on other occasions.

We may extend this process, by performing operations with respect to the quantity (μ). Thus we may operate on any of the integrals we have obtained by such a symbol as $F\left(\frac{d}{d\mu}\right)$, where F is any rational function; and if it is an entire function, we have merely differentiations to perform. If it is a rational fraction, and the factors of the denominator are real and unequal, we may decompose it into simple rational fractions, each of which may, in its turn, be transformed into a simple integral. If we apply this operation to any of the results we have obtained, we immediately have a definite integral $\int\int..P\varepsilon^{\mu Q}F(Q)\,dv\ldots\,d\theta\ldots$ expressed in a series of single integrals, where the integrations are performed with respect to (μ), and (μ) may be taken between any limits. But (μ) must in no case pass through zero, as the definite integrals, on which we operate with respect to (μ), cannot be found for that value of μ by the processes we have been investigating. There are many other operations of a similar nature, which it is easy to imagine.

I am now come to the second part of this memoir, the investigation of those new methods of summation, and of the definite integrals corresponding to them, to which I have before alluded. Let us consider the series

$$1+\frac{x}{\beta}+\frac{x^2}{\beta(\beta+1).1.2}+\frac{x^3}{\beta(\beta+1)(\beta+2).1.2.3}+\&c.,$$

where (β) is an integer. The following integral is known:

$$\int_0^\pi d\theta\,\varepsilon^{a\cos\theta}\cos(a\sin\theta)\cos n\theta=\frac{\pi}{2}\cdot\frac{a^n}{1.2.3\ldots n};$$

$$\therefore\quad \frac{1}{\Gamma\beta}=\frac{1}{\pi a^{\beta-1}}\int_{-\pi}^{\pi}d\theta\,\varepsilon^{a\cos\theta}\cos(a\sin\theta)\,\varepsilon^{(\beta-1)i\theta}.$$

Hence we find for the sum of the above series,

$$\frac{\Gamma\beta}{\pi a^{\beta-1}}\int_{-\pi}^{\pi}d\theta\,\varepsilon^{a\cos\theta}\cos(a\sin\theta)\,\varepsilon^{(\beta-1)i\theta}\,\varepsilon^{\frac{x\varepsilon^{i\theta}}{a}}.$$

Next let us consider the same series when (β) is a fraction. We have

$$\frac{\Gamma(\beta-1)\,\Gamma(n+1)}{\Gamma(\beta+n)}=\int_0^1 dv\,v^n(1-v)^{\beta-2};$$

$$\therefore\quad \frac{\Gamma\beta}{\Gamma(\beta+n)}=\frac{\beta-1}{\pi a^n}\int_0^1\int_{-\pi}^{\pi}d\theta dv\,v^n(1-v)^{\beta-2}\varepsilon^{a\cos\theta}\cos(a\sin\theta)\,\varepsilon^{ni\theta},$$

except for $n=0$, when

$$\frac{2\Gamma\beta}{\Gamma\beta}=\frac{\beta-1}{\pi}\int_0^1\int_{-\pi}^{\pi}d\theta dv(1-v)^{\beta-2}\varepsilon^{a\cos\theta}\cos(a\sin\theta);$$

and we find for the sum of the series,

$$\frac{\beta-1}{\pi}\int_0^1\int_{-\pi}^{\pi}(1-v)^{\beta-2}\varepsilon^{a\cos\theta}\cos(a\sin\theta)\,\varepsilon^{\frac{vx\varepsilon^{i\theta}}{a}}d\theta dv-1.$$

The following are instances of the application of this method obtained by using series I., III., IV.:—

$$\int_0^1\int_{-\pi}^{\pi} d\theta dz(1-z)^{\frac{1}{3}}\, \varepsilon^{\mu(\alpha+z)\cos\theta}\cos(a\mu\sin\theta)\cos(\mu z\sin\theta)$$

$$=\frac{2\pi}{3}+\frac{\pi}{8\mu^3\alpha^{\frac{3}{2}}}\left\{(2\mu\sqrt{\alpha}-1)\,\varepsilon^{2\mu\sqrt{\alpha}}+(2\mu\sqrt{\alpha}+1)\,\varepsilon^{-2\mu\sqrt{\alpha}}\right\}$$

$$\int_0^1\int_0^1\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varepsilon^{\alpha\cos\theta+\beta\cos\phi+\mu vz\cos(\theta+\phi)}(1-v)^{\frac{1}{3}}(1-z)^{\frac{2}{3}}$$

$$\cos(\alpha\sin\theta)\cos(\beta\sin\phi)\cos\left(\mu vz\sin(\theta+\phi)\right)d\theta d\phi dvdz$$

$$=\frac{27\pi^2}{20}+\frac{2\pi^2}{81(\mu\alpha\beta)^{\frac{5}{3}}}(3\sqrt[3]{\mu\alpha\beta}-2)\,\varepsilon^{3\sqrt[3]{\mu\alpha\beta}}-\frac{2\pi^2}{81(\mu\alpha\beta)^{\frac{5}{3}}}\varepsilon^{\frac{-3\sqrt[3]{\mu\alpha\beta}}{2}}$$

$$\left\{(3\sqrt[3]{\mu\alpha\beta}-2)\cos\frac{3\sqrt{3}}{2}\sqrt[3]{\mu\alpha\beta}-\sqrt{3}\,(3\sqrt[3]{\mu\alpha\beta}+2)\sin\frac{3\sqrt{3}}{2}\sqrt[3]{\mu\alpha\beta}\right\}$$

$$\int_0^1\int_{-\pi}^{\pi} d\theta\, dv\, v(1-v)^{-\frac{1}{2}}\varepsilon^{(\alpha+\mu v)\cos\theta}\cos(2\theta+\mu v\sin\theta)\cos(\alpha\sin\theta)$$

$$=\frac{\pi}{\mu^2}+\frac{\pi}{2\mu^2}(\sqrt{\alpha\mu}-1)\,\varepsilon^{2\sqrt{\alpha\mu}}-\frac{\pi}{2\mu^2}(\sqrt{\alpha\mu}+1)\,\varepsilon^{-2\sqrt{\alpha\mu}}.$$

Again, we know that

$$\int_0^{\frac{\pi}{2}} d\theta\cos^\beta\theta\cos n\theta=\frac{\pi\Gamma(\beta+1)}{2^{\beta+1}\Gamma\left(\frac{\beta+n}{2}+1\right)\Gamma\left(\frac{\beta-n}{2}+1\right)},$$

from which we may deduce the following:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{a+b-2}\theta\,\varepsilon^{(a-b)i\theta}d\theta=\frac{\pi\Gamma(a+b-1)}{2^{a+b-2}\Gamma a\Gamma b}.$$

Now consider the series

$$1+\frac{\alpha}{\beta}x+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\cdot\frac{x^2}{1.2}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\cdot\frac{x^3}{1.2.3}+\&c.,$$

where (α) is greater than β. Then by the above formula

$$\frac{\Gamma(\alpha+n)}{\Gamma(\beta+n)}=\frac{2^{\alpha+n-1}}{\pi}\Gamma(\alpha-\beta+1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\cos^{\alpha+n-1}\theta\,\varepsilon^{(2\beta-\alpha+n-1)i\theta};$$

and we find for the sum of the series,

$$\frac{2^{\alpha-1}}{\pi}\cdot\frac{\Gamma\beta\Gamma(\alpha-\beta+1)}{\Gamma\alpha}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\cos^{\alpha-1}\theta\,\varepsilon^{(2\beta-\alpha-1)i\theta}\,\varepsilon^{2\cos\theta\,\varepsilon^{i\theta}x}.$$

In like manner we can find the sum of the series

$$1+\frac{\alpha}{\beta}\cdot\frac{\alpha'}{\beta'}x+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\cdot\frac{\alpha'(\alpha'+1)}{\beta'(\beta'+1)}\cdot\frac{x^2}{1.2}+\&c.,$$

where α is greater than β, α' than β'.

The use of this integral will give an important extension of the method I have employed for expressing the integrals of differential equations by means of definite integrals. For in order to the success of that method, it is necessary, as is shown in my paper in the Cambridge Mathematical Journal before alluded to, that the magnitude of the factorials (if any) in the numerator of each term of the series to be summed, should be less than that of the corresponding factorials in the denominator; whereas this integral enables us to sum series in which the reverse is the case. I shall now apply the series, whose sum we have just found, to the evaluation of definite integrals, using series VI. and VII. Hence

$$\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta dv\ v(1-v)^2\cos^3\theta\ \varepsilon^{2\mu v\cos^2\theta}\cos(2\mu v\sin\theta\cos\theta+\theta)=\frac{\pi}{4\mu^4}(\mu^2-3\mu+3)\varepsilon^{\mu}-\frac{3\pi}{4\mu^4}+\frac{\pi}{8\mu^2}$$

$$\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta dv\ v(1-v)^3\cos^4\theta\ \varepsilon^{2\mu v\cos^2\theta}\cos(2\mu v\sin\theta\cos\theta+2\theta)$$

$$=\frac{3\pi}{8\mu^5}(\mu-2)^2\varepsilon^{\mu}+\frac{\pi}{8\mu^3}(\mu+3)-\frac{3\pi}{2\mu^5}.$$

By a process similar to those used above, we find

$$1+\frac{2}{\frac{3}{2}.3.1}\frac{\mu^2x^2}{2^2}+\frac{2.3}{\frac{3}{2}.\frac{5}{2}.3.4.1.2}\frac{\mu^4x^4}{2^4}+\&c.$$

$$=-\frac{8}{\mu^4x^4}+\frac{2}{\mu^4x^4}(\mu^2x^2-2\mu x+2)\varepsilon^{\mu x}+\frac{2}{\mu^4x^4}(\mu^2x^2+2\mu x+2)\varepsilon^{-\mu x}.$$

Hence $$\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta d\phi\ \varepsilon^{\alpha\cos\phi+2\mu\cos\theta\cos(\theta+\phi)}\cos\theta\cos(\alpha\sin\phi)\cos 2\big(\phi+\mu\cos\theta\sin(\theta+\phi)\big)$$

$$=-\frac{\pi}{2\mu^2}+\frac{\pi}{4\mu^2}(2\mu\alpha-2\sqrt{\mu\alpha}+1)\,\varepsilon^{2\sqrt{\mu\alpha}}+\frac{\pi}{4\mu^2}(2\mu\alpha+2\sqrt{\mu\alpha}+1)\,\varepsilon^{-2\sqrt{\mu\alpha}}.$$

The following formulæ are found in CRELLE's Journal:—

$$\int_0^{\frac{\pi}{2}}\cos^{a-2}\theta\cot^b\theta\cos a\theta d\theta=\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}\cdot\frac{\pi}{2\cos\frac{b\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}}\cos^{a-2}\theta\cot^b\theta\sin a\theta d\theta=\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}\cdot\frac{\pi}{2\sin\frac{b\pi}{2}};$$

$$\therefore\quad\int_0^{\frac{\pi}{2}}\cos^{a-2}\theta\cot^b\theta\ \varepsilon^{ai\theta}d\theta=\frac{\Gamma(a+b-1)}{\Gamma a\Gamma b}\cdot\frac{\pi}{\sin b\pi}\varepsilon^{i(1-b)\frac{\pi}{2}},$$

whence we find $$\frac{\Gamma(a+b-1)}{\Gamma a}=\frac{\Gamma b\sin b\pi}{\pi}\varepsilon^{-i(1-b)\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos^{a-2}\theta\cot^b\theta\ \varepsilon^{ai\theta}d\theta.$$

In this formula we suppose (b) to be less than unity.

Now put $b=\frac{1}{2}$, then

$$\frac{\Gamma\left(a-\frac{1}{2}\right)}{\Gamma a}=\frac{1}{\sqrt{\pi}}\varepsilon^{-\frac{i\pi}{4}}\int_0^{\frac{\pi}{2}}\cos^{a-2}\theta\cot^{\frac{1}{2}}\theta\,\varepsilon^{\theta i a};$$

and putting $\alpha=n+\frac{5}{2}$, we have

$$\frac{\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)}=\frac{1}{\sqrt{\pi}}\varepsilon^{-\frac{i\pi}{4}}\int_0^{\frac{\pi}{2}}\cos^{n+\frac{1}{2}}\theta\cot^{\frac{1}{2}}\theta\,\varepsilon^{\theta i(n+\frac{5}{2})},$$

whence we find, from series IV.,

$$1+\frac{2}{\frac{5}{2}.3}\mu+\frac{2.3}{\frac{5}{2}.\frac{7}{2}.3.4}\cdot\frac{\mu^2}{1.2}+\text{ \&c.}$$

$$=\frac{3}{2\pi}\varepsilon^{-\frac{i\pi}{4}}\int_0^{\frac{\pi}{2}}\int_{-\pi}^{\pi}\cos^{\frac{1}{2}}\theta\cot^{\frac{1}{2}}\theta\,\varepsilon^{\frac{5\theta i}{2}}\varepsilon^{\cos\phi}\cos\sin\phi\,\varepsilon^{2i\phi}d\phi d\theta\,\varepsilon^{\mu\cos\theta}\,\varepsilon^{i(\theta+\phi)},$$

$\therefore$ $\left(\text{since } \frac{2\pi}{3}\varepsilon^{\frac{i\pi}{4}}=\frac{\pi\sqrt{2}}{3}+\frac{\pi i\sqrt{2}}{3}\right)$ we have

$$\int_0^{\frac{\pi}{2}}\int_{-\pi}^{\pi}\cos^{\frac{1}{2}}\theta\cot^{\frac{1}{2}}\theta\,\varepsilon^{\cos\phi}\cos(\sin\phi)\varepsilon^{\mu\cos\theta\cos(\theta+\phi)}d\phi d\theta\cos\left\{\mu\cos\theta\sin(\theta+\phi)+\frac{5\theta}{2}+2\phi\right\}$$

$$=\frac{\pi}{\mu^2\sqrt{2}}+\frac{\pi}{2\sqrt{2}\mu^2}(\sqrt{\mu}-1)\varepsilon^{2\sqrt{\mu}}-\frac{\pi}{2\sqrt{2}\mu^2}(\sqrt{\mu}+1)\varepsilon^{-2\sqrt{\mu}}.$$

Let us again consider the series

$$1+\frac{\alpha}{\beta}x+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\cdot\frac{x^2}{1.2}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\cdot\frac{x^3}{1.2.3}+\text{\&c.}$$

Then making use of the integrals

$$\Gamma(\alpha+n)=\int_0^{\infty}\varepsilon^{-z}z^{\alpha+n-1}\,dz,\text{ and }\Gamma(\alpha+n)=h^{\alpha+n}\varepsilon^{-\frac{\pi i}{2}(\alpha+n)}\int_0^{\infty}\varepsilon^{hiz}z^{\alpha+n-1}dz,$$

where (h) is a constant quantity, we find as the sum of this series,

$$\frac{\Gamma\beta}{\Gamma\alpha}\cdot\frac{\varepsilon}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}dvdz\,z^{\alpha-1}\varepsilon^{-z}\frac{\varepsilon^{iv}}{(1+iv)^{\beta}}\varepsilon^{\frac{xz}{1+iv}},$$

and

$$\frac{\Gamma\beta}{\Gamma\alpha}\cdot\frac{\varepsilon}{2\pi}h^{\alpha}\varepsilon^{-\frac{\pi i\alpha}{2}}\int_{-0}^{\infty}\int_{-\infty}^{\infty}dvdz\,z^{\alpha-1}\varepsilon^{hiz}\frac{\varepsilon^{iv}}{(1+iv)^{\beta}}\varepsilon^{-\frac{ihzx}{1+iv}};$$

also when β is an integer, we may find the following expressions as the sum of the same series:—

$$\frac{\Gamma\beta}{\Gamma\alpha}\cdot\frac{1}{\pi c^{\beta-1}}\int_0^{\infty}\int_{-\pi}^{\pi}d\theta dz\,z^{\alpha-1}\varepsilon^{-z}\cos(c\sin\theta)\,\varepsilon^{c\cos\theta}.\varepsilon^{(\beta-1)i\theta}.\varepsilon^{\frac{xz\varepsilon^{i\theta}}{c}},$$

and also $$\frac{\Gamma\beta}{\Gamma\alpha}\cdot\frac{1}{\pi c^{\beta-1}}\cdot h^{\alpha}\varepsilon^{-\frac{\pi i\alpha}{2}}\int_0^{\infty}\int_{-\pi}^{\pi}d\theta dz\,z^{\alpha-1}\varepsilon^{hiz}\cos(c\sin\theta)\varepsilon^{c\cos\theta}.\varepsilon^{(\beta-1)i\theta}.\varepsilon^{-\frac{ihzx\varepsilon^{i\theta}}{c}}.$$

From hence we obtain, using the first of the two integrals and the series

$$1+\frac{2}{4}\mu+\frac{2.3}{4.5}\cdot\frac{\mu^2}{1.2}+\frac{2.3.4}{4.5.6}\cdot\frac{\mu^3}{1.2.3}+\&c.=\frac{6}{\mu^3}(\mu-2)\varepsilon^{\mu}+\frac{6}{\mu^3}(\mu+2)$$

$$\int_0^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta dz\varepsilon^{\mu z\cos^2\theta-z}z\cos^2\theta\cos(\mu z\sin\theta\cos\theta+4\theta-\tan\theta)=\frac{2\pi}{\mu^3\varepsilon}(\mu-2)\varepsilon^{\mu}+\frac{2\pi}{\mu^3\varepsilon}(\mu+2),$$

and also $$\int_0^{\infty}\int_{-\pi}^{\pi}d\theta dz\varepsilon^{c\cos\theta+\frac{\mu z\cos\theta}{c}-z}z\cos(c\sin\theta)\cos\left(\frac{\mu z\sin\theta}{c}+3\theta\right)$$

$$=\frac{\pi c^3}{\mu^3}(\mu-2)\varepsilon^{\mu}+\frac{\pi c^3}{\mu^3}(\mu+2).$$

The second integral will require in its applications, that we equate possible and impossible parts, in other respects the results will be analogous to those we have just obtained.

There are one or two other methods of summation which I shall briefly notice.

We see at once that

$$1+\frac{\mu}{2}+\frac{1}{1.2}\cdot\frac{\mu^2}{3}+\frac{1}{1.2.3}\cdot\frac{\mu^3}{4}+\&c.=\frac{\varepsilon^{\mu}-1}{\mu}.$$

Now if (r) be any integer, $\frac{1}{r}=\frac{2(-1)^{r-1}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\log_{\varepsilon}\cos\theta\varepsilon^{2r i\theta}.$

Hence $$1+\frac{\mu}{2}+\frac{1}{1.2}\cdot\frac{\mu^2}{3}+\frac{1}{1.2.3}\cdot\frac{\mu^3}{4}+\&c.=\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\log_{\varepsilon}\cos\theta\,\varepsilon^{2i\theta}.\varepsilon^{-\mu\varepsilon^{2i\theta}}$$

Whence $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\log_{\varepsilon}\cos\theta\,\varepsilon^{-\mu\cos 2\theta}\cos(\mu\sin 2\theta-2\theta)=\frac{\pi}{2}\cdot\frac{\varepsilon^{\mu}-1}{\mu}.$$

The integral $$\int_0^{\pi}\theta\sin\theta\cos^{2r}\theta=\frac{\pi}{2r+1}$$

can be employed in the same way.

Again, $$\int_0^{\frac{\pi}{2}}\cos^n\theta\cos n\theta\,d\theta=\frac{\pi}{2}\cdot\frac{1}{2^n},$$

whence $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\cos^n\theta\,\varepsilon^{ni\theta}=\frac{\pi}{2^n}.$$

Hence using the series $$1+\frac{\mu^2}{\frac{1}{2}\cdot 1}\cdot\frac{1}{2^2}+\frac{\mu^4}{\frac{1}{2}\cdot\frac{3}{2}\cdot 1\cdot 2}\cdot\frac{1}{2^4}+\&c.=\frac{\varepsilon^{\mu}+\varepsilon^{-\mu}}{2},$$

we find

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta d\phi\cos^{-\frac{3}{2}}\phi\,\varepsilon^{\mu^2\cos^2\theta\cos\phi\cos(2\theta-\phi)}\cos\left(\mu^2\sin(2\theta-\phi)\cos^2\theta\cos\phi+\tan\phi-\frac{\phi}{2}\right)$$

$$=\frac{\sqrt{\pi}}{\varepsilon}(\varepsilon^{\mu}+\varepsilon^{-\mu}).$$

There are some other definite integrals which we may use in the summation of factorial series, as

$$\int_0^{\frac{\pi}{2}} d\theta \cos^n\theta \cos n\theta \cos 2r\theta = \frac{\pi}{4} \cdot \frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r} \frac{1}{2^n},$$

$$\int_0^{\pi} \frac{d\theta \sin^{2n}\theta}{(1-2a\cos\theta+a^2)} = \frac{(2n-1)(2n-3)\ldots 3.1}{2n(2n-2)\ldots 4.2} \cdot \frac{\pi}{2},$$

$$\int_0^1 \frac{dx\, x^{\alpha-1}(1-x)^{\beta-1}}{(x+a)^{\alpha+\beta}} = \frac{1}{a^{\beta}(1+a)^{\alpha}} \cdot \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)},$$

$$\int_{-\infty}^{\infty} \frac{dx}{(a+ix)^m(b-ix)^n} = 2\pi(a+b)^{1-m-n} \cdot \frac{1.2.3\ldots m+n-2}{1.2.3\ldots m-1.1.2.3\ldots n-1},$$

$$\int_0^{\infty} \frac{x^{m-\frac{1}{2}}dx}{\{(x+a)(x+b)\}^i} = \frac{\Gamma\frac{1}{2}\Gamma\left(n-\frac{1}{2}\right)}{\Gamma n} \frac{1}{(\sqrt{a}+\sqrt{b})^{2n-1}},$$

and probably some besides.

I shall now offer a few observations on the nature of the integrals we have been discussing. The preceding investigations appear to be equivalent to a solution of the following problem:—"To find the definite integrals, whose values can be determined in finite terms by the solution of linear differential equations with variable coefficients." It should seem that the definite integrals, which we have considered in this paper, are the most general ones of any importance, whose values can be found in this way, for the following reasons:—If we expand any definite integral, which is a solution of a differential equation, and its equivalent in terms of the principal variable, and equate like powers of that variable, we obtain a series of definite integrals of a simpler kind, each equal to a fraction whose numerator and denominator consist of factorials, and can therefore be expressed by the products of Eulerian integrals, or to the sum of such fractions. Now I have employed all the more important definite integrals of this class, which are yet known, in the summation of the series which satisfy the differential equation

$$(ax^n+bx^{n-r})\frac{d^ny}{dx^n}+(a'x^{n-1}+b'x^{n-r-1})\frac{d^{n-1}y}{dx^{n-1}}+\&c.=0;$$

and as the properties of the Eulerian integrals have been much studied, and the integrals whose values are dependent on them consequently well known, it is probable that the definite integrals, which we have considered in this paper, embrace all the more important ones whose values can be determined in finite terms by the solution of the above equation. Were we to employ equations of a more general form, we should find that the successive terms of the series which express their solutions, would be given by equations of finite differences, in which the members equated to zero would each consist of more than two terms. Consequently we should be unable in the general case to sum the resulting series by means of definite integrals; and in those cases in which we might find this possible, the integration of the differential equations in finite terms would be practicable in very few cases. The following method of determining a well-known definite integral is here added, to show the connexion between previous investigations relative to definite integrals, and those given in the present memoir.

We know that $1-r^2+\frac{r^4}{1.2}-\frac{r^6}{1.2.3}+\cdots=\varepsilon^{-r^2}$,

or $$1-\frac{(2r)^2}{1.2}\cdot\frac{1}{2}+\frac{(2r)^4}{1.2.3.4}\cdot\frac{1.3}{2^2}-\frac{(2r)^6}{1.2.3.4.5.6}\cdot\frac{1.3.5}{2^3}+\&c.=\varepsilon^{-r^2}.$$

Hence remembering that $\int_0^\infty dz\, z^{2n}\varepsilon^{-z^2}=\frac{1.3..2n-1}{2^n}\cdot\frac{\sqrt{\pi}}{2}$,

we find $$\int_0^\infty \varepsilon^{-z^2}\cos 2rz=\frac{\sqrt{\pi}}{2}\varepsilon^{-r^2}.$$

I shall now enter on some investigations connected with LAGRANGE's theorem.

Let $1-y+\alpha y^r=0$ be an algebraical equation. Then LAGRANGE's theorem gives us the following series:—

$$y^m=1+m\alpha+\frac{m(m+2r-1)}{1.2}\alpha^2+\&c.+\frac{m(m+nr-1)\ (m+nr-2)\ldots.(m+n(r-1)+1)}{1.2.3\ldots n}\alpha^n+\&c.$$

If we apply the usual test of convergency to this series, we find that $(r-1)\alpha$ must be less than unity.

Then we see that

$$\frac{1}{m}\cdot\frac{dy^m}{d\alpha}=1+(m+2r-1)\alpha+\frac{(m+3r-1)\ (m+3r-2)}{1.2}\alpha^2+\&c.$$
$$+\frac{(m+nr-1)\ (m+nr-2)\ldots(m+n(r-1)+1)}{1.2.3\ldots(n-1)}\alpha^{n-1}+\&c.$$

Now $$(m+nr-1)\ (m+nr-2)\ldots(m+n(r-1)+1)=\frac{\Gamma(m+nr)}{\Gamma(m+n(r-1)+1)};$$

wherefore, since $$\frac{\Gamma(a+b-1)}{\Gamma a\Gamma b}=\frac{2^{a+b-2}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{a+b-2}\theta\ \varepsilon^{(a-b)i\theta}d\theta,$$

we have $$(m+nr-1)\ldots(m+n(r-1)+1)$$
$$=\frac{2^{m+nr-1}\Gamma(n)}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{m+nr-1}\theta\ \varepsilon^{(m+n(r-2)+1)i\theta}d\theta$$
$$=\frac{2^{m+nr-1}}{\pi}\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta dz\cos^{m+nr-1}\theta\ \varepsilon^{(m+n(r-2)+1)i\theta}.z^{n-1}\varepsilon^{-z}.$$

Hence we have $$1+(m+2r-1)\alpha+\frac{(m+3r-1)\ (m+3r-2)}{1.2}\alpha^2+\&c.$$
$$=\frac{2^{m+r-1}}{\pi}\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta dz\cos^{m+r-1}\theta\ \varepsilon^{2^r\alpha z\cos^r\theta\cos(r-2)\theta-z}$$
$$\cos(2^r az\cos^r\theta\sin(r-2)\theta+(m+r-1)\theta);$$

$$\therefore \quad \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta dz \cos^{m+r-1}\theta \, \varepsilon^{2^r \alpha z \cos^r\theta \cos(r-2)\theta - z}$$

$$\cos(2^r\alpha z \cos^r\theta \sin(r-2)\theta + (m+r-1)\theta) = \frac{\pi}{2^{m+r-1}m} \cdot \frac{d.y^m}{da}.$$

Let $r=2$, then we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos^{m+1}\theta \cos(m+1)\theta}{1-c\cos^2\theta} = \frac{2\pi}{m} \cdot \frac{d}{dc}\left\{\frac{1-\sqrt{1-c}}{c}\right\}^m,$$

where (c) is of course less than unity; an integral given by ABEL.

When $2^r\alpha$ is less than unity we can always integrate with respect to (z), but may obtain a single integral more simply by proceeding as follows:—

We have
$$\frac{(m+nr-1)\ (m+nr-2)\ldots(m+n(r-1)+1)}{1.2.3\ldots n-1}$$

$$= \frac{2^{m+nr-1}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cos^{m+nr-1}\theta \, \varepsilon^{(m+n(r-2)+1)i\theta};$$

consequently we find by summing a geometrical progression,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cos^{m+r-1}\theta \left\{\frac{\cos(m+r-1)\theta - 2^r a \cos^r\theta \cos(m+1)\theta}{1-2^{r+1}a\cos^r\theta\cos(r-2)\theta + 2^{2r}a^2\cos^{2r}\theta}\right\} = \frac{\pi}{2^{m+r-1}m} \frac{dy^m}{da}.$$

When $r=2$ this result coincides with that last obtained. We may obtain a very general result by applying FOURIER'S theorem to the series of LAGRANGE and LAPLACE as follows:—

If $u=f(y)$, and $y=z+x\phi(y)$,

we have
$$u=f(z)+\{\phi(z)f'(z)\}x+\frac{d}{dz}\{\phi^2 z f' z\}\frac{x^2}{1.2}+\&c.;$$

$$\therefore \quad \frac{du}{dx}=\phi(z)f'(z)+\frac{d}{dz}\{\phi^2(z)f'z\}x+\frac{d^2}{dz^2}\{\phi^3(z)f'(z)\}\frac{x^2}{1.2}+\&c.;$$

Now we generally have
$$F(z)=\int_{-\infty}^\infty\int_{-\infty}^\infty \cos\alpha(z-z')Fz'\frac{d\alpha.dz'}{2\pi},$$

whence
$$\phi^n(z)f'z=\int_{-\infty}^\infty\int_{-\infty}^\infty \varepsilon^{i\alpha(z-z')}\phi^n(z')f'(z')\frac{d\alpha.dz'}{2\pi}$$

and
$$\frac{d^{n-1}}{dz^{n-1}}\phi^n(z)f'(z)=\int_{-\infty}^\infty\int_{-\infty}^\infty \varepsilon^{i\alpha(z-z')}(i\alpha)^{n-1}\phi^n(z')f'(z')\frac{d\alpha.dz'}{2\pi}.$$

Hence substituting in the above series, we find

$$\frac{du}{dx}=\int_{-\infty}^\infty\int_{-\infty}^\infty \varepsilon^{i\alpha(z-z')}\phi(z')f'(z')\varepsilon^{i\alpha\phi(z')x}\frac{d\alpha dz'}{2\pi}.$$

Consequently we find the following definite integral:

$$\int_{-\infty}^\infty\int_{-\infty}^\infty d\alpha dz \phi(z')f'(z')\cos\alpha\big(z-z'+x\phi(z')\big)=2\pi\frac{du}{dx}.$$

Again, from LAPLACE's theorem, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\alpha dz' \cos\alpha(z-z'+x\phi_1\phi_2 z')\phi_1\phi_2 z' f'\phi_2 z' = 2\pi\frac{du}{dx},$$

where $u=f(y),\ y=\phi_2(z+x\phi_1 y)$.

These theorems of course suppose the series from whence they were derived to be convergent.

As examples we may take the following.

Let $y=1+xy^3$,

then
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\alpha dz' \cos\alpha(1-z+xz^3)z^3$$
$$=2\pi\frac{d}{dx}\left\{\sqrt[3]{\left(\frac{1}{2x}+\sqrt{\left(\frac{1}{4x^2}-\frac{1}{27x^3}\right)}\right)}+\sqrt[3]{\left(\frac{1}{2x}-\sqrt{\left(\frac{1}{4x^2}-\frac{1}{27x^3}\right)}\right)}\right\}.$$

Also let $y=1+x\varepsilon^y$,

then
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\alpha(1-z'+x\varepsilon^{z'})\varepsilon^z d\alpha dz = \frac{2\pi\varepsilon^y}{1-x\varepsilon^y},$$

which we may modify thus; by eliminating (x)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\alpha\left(\frac{(y-1)\varepsilon^z-(z-1)\varepsilon^y}{\varepsilon^y}\right)\varepsilon^z d\alpha dz = \frac{2\pi\varepsilon^y}{2-y}.$$

Analogous methods apply to series involving BERNOUILLI's numbers; thus we have

$$\frac{x}{\varepsilon^x-1}=1-\frac{x}{2}+\frac{B_1}{1.2}x^2-\frac{B_3}{1.2.3.4}\cdot x^4+\text{\&c.}$$

$$\frac{B_{2n-1}}{\Gamma(2n+1)}=\frac{1}{2^{2n-1}\pi^{2n}}\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\text{\&c.}\right)$$

$$=\frac{1}{2^{2n-1}\pi^{2n}\Gamma 2n}\int_0^1\frac{\left(\log_\varepsilon\frac{1}{z}\right)^{2n-1}dz}{1-z};$$

$$\therefore\quad \frac{x}{\varepsilon^x-1}+\frac{x}{2}-1=\frac{x}{\pi}\int_0^1\frac{dz\sin\left(\frac{x}{2\pi}\log_\varepsilon\frac{1}{z}\right)}{1-z}.$$

Hence we have
$$\int_0^1\frac{\sin(\alpha\log_\varepsilon z)\,dz}{z-1}=\frac{\pi}{2}\cdot\frac{\varepsilon^{2\alpha\pi}+1}{\varepsilon^{2\alpha\pi}-1}-\frac{1}{2\alpha}.$$

In this formula (α) must lie between 0 and 1, as it is necessary for the convergence of the above series that x should be less than 2π.

I now enter upon the consideration of the processes I have before mentioned for reducing multiple integrals to single ones. We easily see the truth of the following equation:—

$$1+\frac{\mu}{\frac{1}{2}.1^2.2^2}+\frac{\mu^2}{\frac{1}{2}.\frac{3}{2}.1^2.2^2.2^4}+\frac{\mu^3}{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.1^2.2^2.3^2.2^6}+\text{\&c.}$$

$$=2+\frac{\mu}{1.2.1}+\frac{\mu^2}{1.2.3.4.1.2}+\frac{\mu^3}{1.2.3.4.5.6.1.2.3}+\text{\&c.}-1.$$

Hence we have
$$\frac{\Gamma\frac{1}{2}.\varepsilon}{2\pi}.\frac{\Gamma.1\varepsilon}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\varepsilon^{i(z+z')}dzdz'}{(1+iz)^{\frac{1}{2}}(1+iz)}\varepsilon^{\frac{\mu}{(1+iz)(1+iz').2^2}}$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}d\theta\varepsilon^{\alpha\cos\theta}\cos(\alpha\sin\theta)\,\varepsilon^{\frac{\epsilon^{2i\theta}}{\alpha^2}.\mu}-1.$$

Hence
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varepsilon^{\frac{\mu}{4}\cos\theta\cos\phi\cos(\theta+\phi)}\cos^{-\frac{3}{2}}\theta\cos^{-1}\phi.d\theta d\phi,$$
$$\cos\left(\frac{\mu}{4}\cos\theta\cos\phi\sin(\theta+\phi)+\frac{\theta}{2}+\phi-(\tan\theta+\tan\phi)\right)$$
$$=\frac{4\sqrt{\pi}}{\varepsilon^2}\int_{-\pi}^{\pi}d\theta\varepsilon^{\alpha\cos\theta}\cos\alpha\sin\theta.\varepsilon^{\frac{\mu\cos 2\theta}{\alpha^2}}\cos\frac{\mu\sin 2\theta}{\alpha^2}-\frac{4\pi^{\frac{3}{2}}}{\varepsilon^2}.\quad\dots\dots\quad(A.)$$

But we may effect these reductions systematically by means of the following proposition due to M. SMAASEN:—

If $$a_0+a_1x+a_2x^2+a_3x^3+\&c.=\phi_1(x),$$
and $$b_0+b_1x+b_2x^2+b_3x^3+\&c.=\phi_2(x),$$
then $$a_0b_0+a_1b_1x+a_2b_2x^2+\&c.$$
$$=\frac{1}{2\pi}\int_0^{\pi}d\theta\{(\phi_1(x\varepsilon^{i\theta})+\phi_1(x\varepsilon^{-i\theta}))(\phi_2(\varepsilon^{i\theta})+\phi_2(\varepsilon^{-i\theta}))\}.$$

M. SMAASEN has also proved in the same paper, that if the sums of the three series
$$a_0+a_1x+a_2x^2+a_3x^3+\&c.$$
$$b_0+b_1x+b_2x^2+b_3x^3+\&c.$$
$$c_0+c_1x+c_2x^2+c_3x^3+\&c.$$
are known, we may determine the sum of the series
$$a_0b_0c_0+a_1b_1c_1x+a_2b_2c_2x^2+\&c.$$
by means of a double integral, but we shall not want this in what follows.

Now
$$1+\frac{x}{1.2}+\frac{x^2}{1.2.3.4}+\frac{x^3}{1.2.3.4.5.6}+\&c.=\frac{\varepsilon^{\sqrt{x}}+\varepsilon^{-\sqrt{x}}}{2}$$
$$1+\mu x+\frac{\mu^2x^2}{1.2}+\frac{\mu^3x^3}{1.2.3}+\&c.=\varepsilon^{\mu x};$$
consequently
$$1+\frac{\mu x}{\frac{1}{2}.1^2.2^2}+\frac{\mu^2x^2}{\frac{1}{2}.\frac{3}{2}.1^2.2^2.2^4}+\&c.$$
$$=\frac{1}{2\pi}\int_0^{\pi}d\theta\left\{\frac{\varepsilon^{\sqrt{x\varepsilon^{i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{-i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{-i\theta}}}}{2}\right\}\{\varepsilon^{\mu\varepsilon^{i\theta}}+\varepsilon^{\mu\varepsilon^{-i\theta}}\}.$$

Now
$$\varepsilon^{\sqrt{x\varepsilon^{i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{-i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{-i\theta}}}$$
$$=2\varepsilon^{\sqrt{x}\cos\frac{\theta}{2}}\cos\left(\sqrt{x}\sin\frac{\theta}{2}\right)+2\varepsilon^{-\sqrt{x}\cos\frac{\theta}{2}}\cos\left(\sqrt{x}\sin\frac{\theta}{2}\right);$$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varepsilon^{\frac{\mu}{4}\cos\theta\cos\phi\cos\theta+\phi}\cos^{-\frac{3}{2}}\theta\cos^{-1}\phi\, d\theta d\phi,$$

$$\cos\left(\frac{\mu}{4}\cos\theta\cos\phi\sin(\theta+\phi)+\frac{\theta}{2}+\phi-(\tan\theta+\tan\phi)\right)$$

$$=\frac{4\sqrt{\pi}}{\varepsilon^2}\int_0^{\pi} d\theta\left\{(\varepsilon^{\cos\frac{\theta}{2}}+\varepsilon^{-\cos\frac{\theta}{2}})\cos\sin\frac{\theta}{2}\right\}\{\varepsilon^{\mu\cos\theta}\cos(\mu\sin\theta)\} \quad . \; . \; . \quad \text{(B.)}$$

Hence we find, by comparing (A.) with (B.),

$$\int_0^{\pi} d\theta\, \varepsilon^{\mu\cos\theta}\cos(\mu\sin\theta)\left\{2\varepsilon^{\frac{\cos 2\theta}{\mu}}\cos\frac{\sin 2\theta}{\mu}-(\varepsilon^{\cos\frac{\theta}{2}}+\varepsilon^{-\cos\frac{\theta}{2}})\cos\sin\frac{\theta}{2}\right\}=\pi.$$

We have already proved that

$$1+\frac{2}{4}x+\frac{2.3}{4.5}\cdot\frac{x^2}{1.2}+\frac{2.3.4}{4.5.6}\cdot\frac{x^3}{1.2.3}+\&c.$$

$$=\frac{6}{x^3}(x+2)+\frac{6}{x^3}(x-2)\varepsilon^{x}.$$

Hence

$$1+\frac{3}{5}\cdot\frac{x}{2}+\frac{3.4}{5.6}\cdot\frac{x^2}{2.3}+\&c.$$

$$=\frac{12}{x^4}(x+2)+\frac{12}{x^4}(x-2)\varepsilon^{x}-\frac{2}{x},$$

and

$$1+\mu x+\frac{\mu^2x^2}{1.2}+\frac{\mu^3x^3}{1.2.3}+\&c.=\varepsilon^{\mu x}.$$

Consequently the theorem of M. SMAASEN will give us the sum of the series

$$1+\frac{3}{2}\cdot\frac{\mu x}{5.1}+\frac{3.4}{2.3}\cdot\frac{\mu^2x^2}{5.6.1.2}+\frac{3.4.5}{2.3.4}\cdot\frac{\mu^3x^3}{5.6.7.1.2.3}+\&c.$$

by means of a single integral, and we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta d\phi\, \varepsilon^{2\mu\cos\theta\cos\phi\cos(\theta-\phi)}\cos^2\theta\cos^3\phi\cos\{2\mu\cos\theta\cos\phi\sin(\theta-\phi)+\tan\phi-5\phi\}$$

$$=\frac{\pi}{6\varepsilon}\int_0^{\pi} d\theta\{6(\cos 3\theta+2\cos 4\theta)+6\varepsilon^{\cos\theta}\cos(3\theta-\sin\theta)$$

$$-12\varepsilon^{\cos\theta}\cos(4\theta-\sin\theta)-\cos\theta\}\varepsilon^{\mu\cos\theta}\cos(\mu\sin\theta).$$

The fundamental idea of the preceding calculations, as will be readily seen, is as follows: to reduce every term of the series proposed to be summed by means of definite integrals to the form of the general term of the series whose sum is given by the common exponential theorem, and then to find the sum of the whole quantity contained under the signs of integration by means of that theorem. The factorials in the numerator of each term may be taken in any order we please relative to those of the denominator, provided that the same relative order is observed in every term throughout the whole series; moreover, we may use different integrals to express the same factorials, so that we can deduce the value of many definite integrals from one series.

I shall now give an example of the summation of a factorial series of a somewhat different nature.

Consider the series—

$$1+\frac{x}{a^2+2^2}+\frac{x^2}{(a^2+2^2)(a^2+4^2)}+\frac{x^3}{(a^2+2^2)(a^2+4^2)\dots(a^2+2^2n^2)}+\&c.,$$

we know that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varepsilon^{a\theta}(\cos\theta)^n=\frac{1.2.4\dots 2n}{(a^2+2^2)(a^2+4^2)\dots(a^2+2^2n^2)}\cdot\frac{\varepsilon^{\frac{a\pi}{2}}-\varepsilon^{-\frac{a\pi}{2}}}{a}.$

Hence by substitution the above series becomes

$$\frac{a}{\varepsilon^{\frac{a\pi}{2}}-\varepsilon^{-\frac{a\pi}{2}}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\varepsilon^{a\theta}\left\{1+\frac{x\cos^2\theta}{1.2}+\frac{x^2\cos^4\theta}{1.2.3.4}+\&c.\right\}$$

$$=\frac{a}{2\left(\varepsilon^{\frac{a\pi}{2}}-\varepsilon^{-\frac{a\pi}{2}}\right)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\varepsilon^{a\theta}\{\varepsilon^{\sqrt{x}\cos\theta}+\varepsilon^{-\sqrt{x}\cos\theta}\}.$$

There are other series of an analogous nature which may be summed in a similar manner: the object of introducing the above summation in this paper, is to point out the use of the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varepsilon^{a\theta}(\cos\theta)^n$, when impossible factors occur in the denominators of the successive terms of a factorial series.

In the 'Exercices de Mathématiques,' CAUCHY has proved that if z be a quantity of the form $\rho(\cos\phi+i\sin\phi)$, and $z\phi(z)$ continually approach zero as ρ indefinitely increases whatever be ϕ, then the residue of $\phi(z)$ is equal to zero, the limits of ρ being 0 and (∞), and those of ϕ, π and $-\pi$. From this theorem he deduces the sums of certain series, which I shall presently consider; but must first give certain results which will be useful in the sequel.

Since $$\int_0^\infty\varepsilon^{-ax^2}\cos 2x dx=\frac{\sqrt{\pi}}{2\sqrt{a}}\varepsilon^{-\frac{1}{a}}$$

$$\therefore\quad \varepsilon^{-\frac{1}{a}}=\frac{\sqrt{a}}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\varepsilon^{-\frac{ax^2}{4}}\cos x dx.$$

Again, since $$\int_{-\infty}^{\infty}\varepsilon^{x-ax^2}=\frac{\sqrt{\pi}}{\sqrt{a}}\varepsilon^{\frac{1}{4a}},$$

we find $$\varepsilon^{\frac{1}{a}}=\frac{\sqrt{a}}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\varepsilon^{x-\frac{ax^2}{4}}dx,$$

whence we have $$\varepsilon^{\frac{1}{a}}-\varepsilon^{-\frac{1}{a}}=\frac{\sqrt{a}}{2\sqrt{\pi}}\int_{-\infty}^{\infty}dx(\varepsilon^x-\cos x)\varepsilon^{-\frac{ax^2}{4}}.$$

The first series we propose to consider is the following:—

$$\frac{1}{x^2-\frac{1}{x^2}}\tan\frac{\pi x^2}{2}+\frac{\frac{1}{3}}{\frac{x^2}{9}-\frac{9}{x^2}}\tan\frac{\pi x^2}{6}+\frac{\frac{1}{5}}{\frac{x^2}{25}-\frac{25}{x^2}}\tan\frac{\pi x^2}{10}+\&c.=\frac{\pi}{16}\left\{\left(\frac{\varepsilon^{\frac{\pi x}{2}}-\varepsilon^{-\frac{\pi x}{2}}}{\varepsilon^{\frac{\pi x}{2}}+\varepsilon^{-\frac{\pi x}{2}}}\right)^2-\tan^2\frac{\pi x}{2}\right\}.$$

Put $\pi x^2=\varrho$, then this series with its sign changed may be resolved into the three following:—

$$\frac{x^2}{2(1+x^2)}\tan\frac{\varrho}{2}+\frac{x^2}{2(9+x^2)}\cdot\frac{1}{3}\tan\frac{\varrho}{3.2}+\frac{x^2}{2(25+x^2)}\frac{1}{5}\tan\frac{\varrho}{5.2}+\&c.$$

$$\frac{x}{4(1-x)}\tan\frac{\varrho}{2}+\frac{x}{4(3-x)}\frac{1}{3}\tan\frac{\varrho}{3.2}+\frac{x}{4(5-x)}\cdot\frac{1}{5}\tan\frac{\varrho}{5.2}+\&c.$$

$$-\frac{x}{4(1+x)}\tan\frac{\varrho}{2}-\frac{x}{4(3+x)}\cdot\frac{1}{3}\tan\frac{\varrho}{3.2}-\frac{x}{4(5+x)}\cdot\frac{1}{5}\cdot\tan\frac{\varrho}{5.2}+\&c.$$

The general terms of these series are respectively,

$$\frac{x^2}{2\{(2n+1)^2+x^2\}}\cdot\frac{1}{2n+1}\tan\frac{\varrho}{2(2n+1)}$$

$$\frac{x}{4\{(2n+1)-x\}}\cdot\frac{1}{2n+1}\tan\frac{\varrho}{2(2n+1)}$$

$$-\frac{x}{4\{(2n+1)-x\}}\cdot\frac{1}{2n+1}\tan\frac{\varrho}{2(2n+1)}.$$

These terms become after transformation, since

$$\int_0^\infty\frac{dz(\varepsilon^{\alpha z}-\varepsilon^{-\alpha z})}{\varepsilon^{\pi z}-\varepsilon^{-\pi z}}=\frac{1}{2}\tan\frac{\alpha}{2},$$

$$\frac{x}{2}\int_0^\infty du\,\varepsilon^{-(2n+1)u}\sin xu\int_0^\infty\frac{dz}{(\varepsilon^{\pi z}-\varepsilon^{-\pi z})\sqrt{\pi\varrho z}}\int_{-\infty}^\infty dv(\varepsilon^v-\cos v)\varepsilon^{-\frac{(2n+1)v^2}{4\rho z}}\int_0^1\frac{ds}{\sqrt{\pi}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}.s^{2n}$$

$$+\frac{x}{4}\int_0^\infty du\,\varepsilon^{-(2n+1-x)u}\int_0^\infty\frac{dz}{(\varepsilon^{\pi z}-\varepsilon^{-\pi z})\sqrt{\pi\rho z}}\int_{-\infty}^\infty dv(\varepsilon^v-\cos v)\varepsilon^{-\frac{(2n+1)v^2}{4\rho z}}\int_0^1\frac{ds}{\sqrt{\pi}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}.s^{2n}$$

$$-\frac{x}{4}\int_0^\infty du\,\varepsilon^{-(2n+1+x)u}\int_0^\infty\frac{dz}{(\varepsilon^{\pi z}-\varepsilon^{-\pi z})\sqrt{\pi\varrho z}}\int_{-\infty}^\infty dv(\varepsilon^v-\cos v)\varepsilon^{-\frac{(2n+1)v^2}{4\rho z}}\int_0^1\frac{ds}{\sqrt{\pi}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}.s^{2n}.$$

Each of the series is consequently reduced to a geometrical progression; wherefore, summing the three progressions and taking the aggregate, we have

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_0^1\frac{(\varepsilon^v-\cos v)\varepsilon^{-u-\frac{v^2}{4\pi x^2 z}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}(2\sin xu+\varepsilon^{xu}-\varepsilon^{-xu})dsdvdudz}{(\varepsilon^{\pi z}-\varepsilon^{-\pi z})\sqrt{z}(1-\varepsilon^{-2u-\frac{v^2}{2\pi x^2 z}}s^2)}$$

$$=\frac{\pi^2}{4}\left\{\tan^2\frac{\pi x}{2}-\left(\frac{\varepsilon^{\frac{\pi x}{2}}-\varepsilon^{-\frac{\pi x}{2}}}{\varepsilon^{\frac{\pi x}{2}}+\varepsilon^{-\frac{\pi x}{2}}}\right)^2\right\}.$$

Next, let us consider the series

$$\frac{1}{x^2-\frac{1}{x^2}}\cot\pi x^2+\frac{\frac{1}{2}}{\frac{x^2}{4}-\frac{4}{x^2}}\cot\frac{\pi x^2}{2}+\frac{\frac{1}{3}}{\frac{x^2}{9}-\frac{9}{x^2}}\cot\frac{\pi x^3}{3}+\&c.$$

$$=\frac{\pi}{8}\left\{\cot^2\pi x-\left(\frac{\varepsilon^{\pi x}+\varepsilon^{-\pi x}}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}}\right)^2\right\}.$$

Let each term of this series be transformed by means of the integral

$$\int_0^\infty \frac{z^{\alpha-1}dz}{1-z}=\pi\cot\alpha\pi,$$

and we have

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_0^1 \frac{\varepsilon^{-u+v-\frac{v^2}{4x^2\log_\varepsilon z}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}\,(2\sin xu+\varepsilon^{xu}-\varepsilon^{-xu})\,dsdvdudz}{(z-z^2)\log_\varepsilon^{\frac{1}{2}}z\,(1-\varepsilon^{-u-\frac{v^2}{4x^2\log_\varepsilon z}}s)}$$

$$=\pi^{\frac{5}{2}}\left\{\left(\frac{\varepsilon^{\pi x}+\varepsilon^{-\pi x}}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}}\right)^2-\cot^2\pi x\right\}.$$

Again, $$\frac{1}{x^2-\frac{1}{x^2}}\sec\frac{\pi x^2}{2}-\frac{\frac{1}{3}}{\frac{x^2}{9}-\frac{9}{x^2}}\sec\frac{\pi x^2}{6}+\frac{\frac{1}{5}}{\frac{x^2}{25}-\frac{25}{x^2}}\sec\frac{\pi x^2}{10}-\&c.$$

$$=\frac{\pi}{16}\left\{\left(\frac{2}{\varepsilon^{\frac{\pi x}{2}}+\varepsilon^{-\frac{\pi x}{2}}}\right)^2-\sec^2\frac{\pi x}{2}\right\}.$$

Here we reduce each term by means of the integral

$$\int_0^\infty \frac{dz(\varepsilon^{\alpha z}+\varepsilon^{-\alpha z})}{\varepsilon^{\pi z}+\varepsilon^{-\pi z}}=\frac{1}{2}\sec\frac{\alpha}{2};$$

and we have

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_0^1 \frac{(\varepsilon^v+\cos v)\,\varepsilon^{-u-\frac{v^2}{4\pi x^2 z}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}\,(2\sin xu+\varepsilon^{xu}-\varepsilon^{-xu})\,dsdvdudz}{(\varepsilon^{\pi z}+\varepsilon^{-\pi z})\sqrt{z}\,(1+\varepsilon^{-2u-\frac{v^2}{2\pi x^2 z}}s^2)}$$

$$=\frac{\pi^2}{4}\left\{\sec^2\frac{\pi x}{2}-\left(\frac{2}{\varepsilon^{\frac{\pi x}{2}}+\varepsilon^{-\frac{\pi x}{2}}}\right)^2\right\}.$$

Also, since $$\frac{1}{x^2-\frac{1}{x^2}}\operatorname{cosec}\pi x^2-\frac{\frac{1}{2}}{\frac{x^2}{4}-\frac{4}{x^2}}\operatorname{cosec}\frac{\pi x^2}{2}+\frac{\frac{1}{3}}{\frac{x^2}{9}-\frac{9}{x^2}}\operatorname{cosec}\frac{\pi x^2}{3}-\&c.$$

$$=\frac{\pi}{8}\left\{\left(\frac{2}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}}\right)^2-\operatorname{cosec}^2\pi x\right\},$$

we have, transforming each term of the series by means of the integral

$$\int_0^\infty \frac{z^{\alpha-1}dz}{1+z}=\pi\operatorname{cosec}\alpha\pi,$$

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^1 \frac{\varepsilon^{-u+v-\frac{v^2}{4x^2\log_\varepsilon z}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}\,(2\sin xu+\varepsilon^{xu}-\varepsilon^{-xu})\,dsdvdudz}{(z+z^2)\log_\varepsilon^{\frac{1}{2}} z(1+\varepsilon^{-u-\frac{v^2}{4x^2\log_\varepsilon z}}s)}$$

$$=\pi^{\frac{5}{2}}\left\{\operatorname{cosec}^2\pi x-\left(\frac{2}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}}\right)^2\right\}.$$

Let us next consider the series

$$\frac{\sin\theta}{1+\alpha^2}+\frac{2\sin 2\theta}{2^2+\alpha^2}+\frac{3\sin 3\theta}{3^2+\alpha^2}+\&c.=\frac{\pi}{2}\cdot\frac{\varepsilon^{\alpha(\pi-\theta)}-\varepsilon^{-\alpha(\pi-\theta)}}{\varepsilon^{\alpha\pi}-\varepsilon^{-\alpha\pi}}.$$

The general term of this series is $\frac{n\sin n\theta}{n^2+\alpha^2}$;

also, we have
$$\frac{n}{n^2+\alpha^2}=\int_0^\infty \varepsilon^{-nz}\sin\alpha z dz,$$

and
$$\varepsilon^{-z}\sin\theta+\varepsilon^{-2z}\sin 2\theta+\varepsilon^{-3z}\sin 3\theta+\&c.=\frac{\varepsilon^{-z}\sin\theta}{1-2\varepsilon^{-z}\cos\theta+\varepsilon^{-2z}}.$$

Hence
$$\int_0^\infty \frac{\varepsilon^{-z}\sin\alpha z dz}{1-2\varepsilon^{-z}\cos\theta+\varepsilon^{-2z}}=\frac{\pi}{2\sin\theta}\cdot\frac{\varepsilon^{\alpha(\pi-\theta)}-\varepsilon^{-\alpha(\pi-\theta)}}{\varepsilon^{\alpha\pi}-\varepsilon^{-\alpha\pi}}.$$

In like manner from the series

$$\frac{\cos\theta}{1+\alpha^2}+\frac{\cos 2\theta}{2^2+\alpha^2}+\frac{\cos 3\theta}{3^2+\alpha^2}+\&c.=\frac{\pi}{2\alpha}\cdot\frac{\varepsilon^{\alpha(\pi-\theta)}+\varepsilon^{-\alpha(\pi-\theta)}}{\varepsilon^{\alpha\pi}-\varepsilon^{-\alpha\pi}}-\frac{1}{2\alpha^2}$$

$$\int_0^\infty \cos\alpha z\frac{(\varepsilon^{-z}\cos\theta-\varepsilon^{-2z})\,dz}{1-2\varepsilon^{-z}\cos\theta+\varepsilon^{-2z}}=\frac{\pi}{2}\cdot\frac{\varepsilon^{\alpha(\pi-\theta)}+\varepsilon^{-\alpha(\pi-\theta)}}{\varepsilon^{\alpha\pi}-\varepsilon^{-\alpha\pi}}-\frac{1}{2\alpha}.$$

Let us next consider the series

$$\frac{1}{\varepsilon^{\pi}-\varepsilon^{-\pi}}-\frac{2}{\varepsilon^{2\pi}-\varepsilon^{-2\pi}}+\frac{3}{\varepsilon^{3\pi}-\varepsilon^{-3\pi}}-\&c.=\frac{1}{4\pi}.$$

We know that
$$\int_0^\infty \frac{\sin\mu z.dz}{\varepsilon^{2\pi z}-1}=\frac{1}{4}\cdot\frac{\varepsilon^{\mu}+1}{\varepsilon^{\mu}-1}-\frac{1}{2\mu},$$

$$\therefore\quad \int_0^\infty \frac{\sin 2n\pi z dz}{\varepsilon^{2\pi z}-1}=\frac{1}{2}\,\frac{1}{1-\varepsilon^{-2n\pi}}-\frac{1}{4}-\frac{1}{4n\pi},$$

$$\therefore\quad \frac{1}{\varepsilon^{n\pi}-\varepsilon^{-n\pi}}=2\varepsilon^{-n\pi}\int_0^\infty \frac{\sin 2n\pi z.dz}{\varepsilon^{2\pi z}-1}+\frac{\varepsilon^{-n\pi}}{2}+\frac{\varepsilon^{-n\pi}}{2n\pi}.$$

Now $x\sin\theta-x^2\sin 2\theta+x^3\sin 3\theta-\&c.$,

$$=\frac{x\sin\theta}{1+2x\cos\theta+x^2}.$$

From whence we have $\quad x\sin\theta-2x^2\sin 2\theta+3x^3\sin 3\theta-\&c.$

$$=\frac{x\sin\theta(1-x^2)}{(1+2x\cos\theta+x^2)^2}.$$

It is hence evident that

$$\frac{1}{4\pi}=2\varepsilon^{\pi}(\varepsilon^{2\pi}-1)\int_0^\infty \frac{dz\sin 2\pi z}{(\varepsilon^{2\pi}+2\varepsilon^{\pi}\cos 2\pi z+1)^2(\varepsilon^{2\pi z}-1)}+\frac{1}{2}\,\frac{\varepsilon^{\pi}}{(\varepsilon^{\pi}+1)^2}+\frac{1}{2\pi}\cdot\frac{1}{\varepsilon^{\pi}+1};$$

$$\therefore\quad \int_0^\infty \frac{dz.\sin 2\pi z}{(\varepsilon^{2\pi}+2\varepsilon^{\pi}\cos 2\pi z+1)^2(\varepsilon^{2\pi z}-1)}=\frac{1}{4\varepsilon^{\pi}(\varepsilon^{\pi}+1)^2(\varepsilon^{\pi}-1)}\left\{\frac{\varepsilon^{2\pi}-1}{2\pi}-\varepsilon^{\pi}\right\}.$$

Again, we have $$\frac{1}{\varepsilon^{\pi}-\varepsilon^{-\pi}}-\frac{2^{4m+1}}{\varepsilon^{2\pi}-\varepsilon^{-2\pi}}+\frac{3^{4m+1}}{\varepsilon^{3\pi}-\varepsilon^{-3\pi}}-\&c.=0.$$

We must transform the element n^{4m+1} thus:

$$n^{4m+1}=\frac{1}{\Gamma(4m+1)}\int_0^1 x^{\frac{1}{n}-1}\left(\log_\varepsilon\frac{1}{x}\right)^{4m}dx=\frac{1}{\Gamma(4m+1)}\int_0^1 \varepsilon^{\frac{\log_\varepsilon x}{n}}\frac{1}{x}\left(\log_\varepsilon\frac{1}{x}\right)^{4m}dx.$$

Again, $$\varepsilon^{\frac{\log_\varepsilon x}{n}}=\frac{\sqrt{n}}{2\sqrt{\pi\log_\varepsilon x}}\int_{-\infty}^{\infty}\varepsilon^{z-\frac{nz^2}{4\log_\varepsilon x}}dz.$$

Lastly, $$\sqrt{n}=\frac{2n}{\sqrt{\pi}}\int_0^\infty \varepsilon^{-nv^2}dv.$$

Hence, combining these integrals together, and substituting for $\frac{1}{\varepsilon^{n\pi}-\varepsilon^{-n\pi}}$, as before, we are able to transform the above series into one which can be summed by the ordinary rules. The resulting definite integral will of course be equal to zero.

Cauchy has applied the methods of the residual calculus to the determination of the sum of the series whose general term is

$$(-1)^{n-1}\frac{\varepsilon^{n\alpha}-\varepsilon^{-n\alpha}}{\varepsilon^{n\pi}-\varepsilon^{-n\pi}}\cdot\frac{n\cos n\alpha}{n^4+c^4}$$

in finite terms. We may transform the element $\frac{1}{n^4+c^4}$ thus:

$$\frac{1}{n^4+c^4}=\frac{1}{c^2}\int_0^\infty \varepsilon^{-n^2z^2}\sin c^2z dz.$$

Again, $$\varepsilon^{-n^2z^2}=\frac{2}{\sqrt{\pi z}}\int_0^\infty \varepsilon^{-\frac{v^2}{z^2}}\cos 2nz.$$

Wherefore, combining these integrals, and transforming the other elements as before, we may find its sum by means of definite integrals. We may resolve $\frac{n}{n^4+c^4}$ into its partial fractions, and then find the sum of the series, which would be simpler.

The transformation of $\varepsilon^{-n^2z^2}$ which I have used above, is due to Professor Kummer, who has applied it in the seventeenth volume of Crelle's Journal, in a paper to which I am indebted for many ideas relative to the connexion of definite integrals with series, to the expression of the series

$$1+q+q^4+q^9+\&c.,$$

and others of a similar nature by means of a definite integral. The integral $\int_0^\infty\frac{\sin\mu z.dz}{\varepsilon^{2\pi z}-1}$ was first applied to the summation of series, whose terms involve elements of the form $\frac{1}{\alpha-\beta x^n}$, by Poisson in his Memoir on the Distribution of Electricity in two electrized spheres, which mutually act upon each other. He proves that the calculation of the electrical arrangement depends upon the value of the definite integral,

$$\int_0^\infty \frac{\sin cz.dz}{(\varepsilon^{2\pi z}-1)\,(a+b\sin^2 cz)}.$$

I mention this on account of its analogy with the definite integral

$$\int_0^\infty \frac{dz \sin 2\pi z}{(\varepsilon^{2\pi}+2\varepsilon^{\pi}\cos 2\pi z+1)^2\,(\varepsilon^{2\pi z}-1)},$$

whose value is found above. The principles contained in this paper will enable us at once to find the sums of the series

$$1+x+\frac{x^{\frac{1}{2}}}{1.2}+\frac{x^{\frac{1}{3}}}{1.2.3}+\frac{x^{\frac{1}{4}}}{1.2.3.4}+\&c.$$

$$1+\frac{\tan\theta}{1}+\frac{\tan 2\theta}{1.2}+\frac{\tan 3\theta}{1.2.3}+\&c.$$

$$1+\frac{\sec\theta}{1}+\frac{\sec\frac{\theta}{2}}{1.2}+\frac{\sec\frac{\theta}{3}}{1.2.3}+\&c.$$

and of many others which can be imagined, by means of definite integrals. The definite integral of Poisson given above occurs in the solution of a functional equation; and it is probable that series similar to those I have been discussing in this paper, may be useful in enabling us to express the solutions of other functional equations by definite integrals.